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Electromagnetic properties of a toroidal solenoid

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Abstract. The electromagnetic properties of the toroidal solenoid with an alternating current are discussed. The multipole expansion for the electromagnetic field of the toroidal solenoid is obtained and explicit expressions for the toroidal form factors and moments are given. Concrete realization of the current flowing in the solenoid's winding is presented. It is shown that uniform rotation of the toroidal solenoid with a constant current in its winding leads to the appearance of a magnetic field outside the solenoid. For non-uniform rotation an electric field arises.

1. Introduction

The toroidal solenoid is a unique object exhibiting a number of interesting properties. For example, the magnetic field H may vanish either inside or outside the solenoid depending on the current distribution on the solenoid surface [1–3]. The small value of magnetic flux leakage from the solenoid has favoured its application in controlled thermonuclear physics [4, 5]. As an accumulator of electromagnetic energy, it is extensively used in electromagnetic launcher technology [6, 7].

The toroidal solenoid is an ideal device for both experimental [8] and theoretical [9] investigations of the Aharonov–Bohm effect. According to [10, 11] the current flowing in the winding of a toroidal solenoid is characterized by new kinds of multipole moments: the so-called toroidal (or anapole) multipole moments. They are now an object of extensive theoretical and experimental studies [11–16]. Usually, the electromagnetic field of a toroidal solenoid is obtained either through numerical integration of the Poisson and Helmholtz equations [17] or through their physical simulation [18]. For the static case, closed expressions for the vector potential of the toroidal solenoid were obtained in [19] and their properties were discussed in [3]. In [3] the electromagnetic field of toroidal solenoid with a time-dependent current was considered. Regretfully, only one half page was devoted to the most important case of the periodically varying current. Numerous discussions with radio engineers and physicists dealing with toroidal moments has enabled us to consider this particular case in a more complete and systematic way, emphasizing the principal points and physical consequences.

This paper is organized as follows. In section 2 we present straightforward calculations of vector potential, field strengths and the Poynting vector of the toroidal solenoid with the current periodically changing with time. This is important, for example, for the construction of toroidal radio antennae [20]. There is a lot of controversy concerning toroidal moments in the literature. To clarify this point, we determine in section 3 the usual multipole expansion [21-24] of the electromagnetic field for the toroidal solenoid and find that electric multipole moments exactly coincide with the toroidal ones. There are known [11, 36] conditions under which toroidal moments become non-trivial. The interaction of a toroidal solenoid with an external electromagnetic field is treated in section 4. There exist theoretical considerations [25, 26] of electric fields arising from electron motion relative to the resting positive lattice ions as well as experimental attempts [27] to measure these fields. There has also been experimental research [28] of the electric field arising from the rapid rotation of a toroidal solenoid. In sections 5 and 6 we evaluate electromagnetic field of the rotating toroidal solenoid. It turns out that only a magnetic field appears outside the uniformly rotating (around its symmetry axis) solenoid. An electric field arises in the case of non-uniform solenoid rotation.

2. The approximate electromagnetic field

Consider the torus

$$(\rho - d)^2 + z^2 = R^2. \tag{2.1}$$

Introduce the coordinates $\tilde{R}$, ψ

$$\rho = d + \tilde{R}\cos\psi \qquad z = \tilde{R}\sin\psi. \tag{2.2}$$

The value $\tilde{R} = R$ corresponds to the torus (2.1). Let the periodical poloidal current flow over the torus surface (each particular coil lies in the φ = constant plane). The density of this current is

$$\boldsymbol{j}\cdot\cos\omega t. \tag{2.3}$$

Here

$$\boldsymbol{j} = -\frac{gc}{4\pi}\frac{\delta(\tilde{R} - R)}{d + R\cos\psi}\boldsymbol{n}_\psi \tag{2.4}$$

$g = 2NI/c$, N is the total number of coils in the solenoid's winding, I is the current in a particular coil, $\boldsymbol{n}_\psi$ is the unit vector tangential to the torus surface and lying in the φ = constant plane

$$\boldsymbol{n}_\psi = \boldsymbol{n}_z\cos\psi - (\boldsymbol{n}_x\cos\varphi + \boldsymbol{n}_y\sin\varphi)\sin\psi.$$

In the Coulomb gauge (div $\boldsymbol{A} = 0$) the vector potential (VP) corresponding to the current (2.3) is given by

$$\boldsymbol{A} = \frac{1}{c}\int\frac{1}{|\boldsymbol{r} - \boldsymbol{r}_1|}\boldsymbol{j}(\boldsymbol{r}_1)\cos(\omega t - k|\boldsymbol{r} - \boldsymbol{r}_1|)\,\mathrm{d}V_1 \qquad k = \frac{\omega}{c} \tag{2.5}$$

where the volume element is

$$\mathrm{d}V_1 = \tilde{R}(d + \tilde{R}\cos\psi)\,\mathrm{d}\tilde{R}\,\mathrm{d}\psi\,\mathrm{d}\varphi.$$

Only two spherical components of $\boldsymbol{A}$, A_θ and A_r differ from zero (see [29] for their derivation). They are simplified if the following conditions are fulfilled:

$$kR \ll 1 \qquad R \ll d \qquad d \ll r \qquad \frac{k\,\mathrm{d}R}{r} \ll 1 \tag{2.6}$$

$$A_r = \frac{\Lambda d \cos\theta}{2r^3}\left[(\cos\Omega + kr\sin\Omega)J_0 + \frac{d\sin\theta}{r}(3\sin\Omega - 2kr\cos\Omega)J_1\right]$$

$$A_\theta = \frac{\Lambda\cos\Omega}{4r^3}[d\sin\theta(J_0 - 3J_2) - 2kr^2 J_1] + \frac{\Lambda}{2r^2}\sin\Omega(J_1 - kd\sin\theta J_2) \tag{2.7}$$

($\Omega = kr - \omega t$, $\Lambda = \pi R^2 g$). The argument of the Bessel functions in (2.7) is $kd \sin\theta$. The non-vanishing field strengths are

$$H_\varphi = \frac{\Lambda}{2r^3}[(k^2r^2 - 1)\sin\Omega + kr\cos\Omega]J_1 + \frac{\Lambda kd}{4r^3}\sin\Omega\sin\theta J_0$$

$$E_\theta = \frac{\Lambda k}{2r^2}\cos\Omega(J_1 - kd\sin\theta J_2) - \frac{\Lambda k}{4r^3}\sin\Omega[d\sin\theta(J_0 - 3J_2) - 2kr^2 J_1] \tag{2.8}$$

$$E_r = -\frac{\Lambda kd\cos\theta}{2r^3}(\sin\Omega - kr\cos\Omega)J_0 + \frac{\Lambda kd^2\sin\theta\cos\theta}{2r^4}(3\cos\Omega + 2kr\sin\Omega)J_1.$$

In the wave zone ($kr \gg 1$)

$$E_\theta = H_\varphi = \frac{1}{2r}\Lambda k^2\sin\Omega J_1 \qquad E_r = \frac{1}{2r^2}\Lambda k^2 d\cos\theta\cos\Omega J_0. \tag{2.9}$$

The radial component of the Poynting vector is

$$S_r = \frac{1}{4\pi c}E_\theta H_\varphi = \frac{1}{4\pi c}\left(\frac{\Lambda k^2\sin\Omega}{2r}\right)^2[J_1(kd\sin\theta)]^2. \tag{2.10}$$

The integral energy flow (averaged over the period) is

$$\frac{1}{4c}\left(\frac{\Lambda k^2}{2}\right)^2\int_0^\pi [J_1(kd\sin\theta)]^2\sin\theta\,\mathrm{d}\theta. \tag{2.11}$$

Being related to the square of the total current ($=NI$) this gives the so-called radiation resistance [30]

$$\frac{\pi^2 k^4 R^4}{4c^3}\int_0^\pi (J_1)^2\sin\theta\,\mathrm{d}\theta. \tag{2.12}$$

The integral occurring here can be taken in a closed form for small and large values of kd

$$\int_0^\pi (J_1)^2\sin\theta\,\mathrm{d}\theta = \begin{cases}(kd)^{2/3} & \text{for } kd \ll 1\\ (kd)^{-1} & \text{for } kd \gg 1.\end{cases}$$

Experimental investigations of the toroidal solenoid with alternate current were performed almost half a century ago. Their description may be found in an excellent book [31], published, regretfully, only in Russian.

It turns out that limitations imposed by (2.6) are not too severe. As an example, consider two experiments performed with a toroidal solenoid [31, 28]. In the first of them [31] the typical frequency was of the order 1 MHz, $R \sim 1$ cm, $d \sim 5$ cm. This gives $R/d \sim 0{\cdot}2$, $kR \sim 2\times10^{-4}$, $kd \sim 10^{-3}$. Thus, approximate field strengths (2.8) are adequate for the description of this experiment. The parameters of another experiment [28] were: $\omega \sim 320$ Hz, $d \sim 14$ cm, $R \sim 4$ cm. Again (2.8) can be used for the analysis.

3. The multipole expansion of the electromagnetic field

3.1. The spherical functions expansion

In what follows we shall use the current given by

$$\boldsymbol{j}\exp(-\mathrm{i}\omega t) \tag{3.1}$$

where $\boldsymbol{j}$ is given by (2.4). As all components of electromagnetic potentials and strengths contain the factor $\exp(-\mathrm{i}\omega t)$ it will be omitted in all intermediatory expressions. It should be restored when the time differentiation is performed or in final expressions from which the real part should be taken. The expansion of the vector potential over the states with definite orbital momentum is as follows:

$$\boldsymbol{A}=\frac{4\pi \mathrm{i}k}{c}\sum h_l(kr)Y_l^m(\theta,\varphi)\int g_l(kr')Y_l^{m*}(\theta',\varphi')\boldsymbol{j}(\boldsymbol{r}')\,\mathrm{d}V'. \tag{3.2}$$

Here

$$g_l(x)=\sqrt{\frac{\pi}{2x}}J_{l+1/2}(x) \qquad h_l(x)=\sqrt{\frac{\pi}{2x}}H^{(1)}_{l+1/2}(x)$$

are the spherical Bessel and Hankel functions. The vector potential may be developed over the spherical basis

$$\begin{gathered}\boldsymbol{A}=\sum(-1)^\mu A_\mu\cdot\boldsymbol{n}_{-\mu} \qquad \mu=0,\pm1\\ \boldsymbol{n}_0=\boldsymbol{n}_z \qquad \boldsymbol{n}_{\pm1}=\mp(\boldsymbol{n}_x\pm\mathrm{i}\boldsymbol{n}_y)/\sqrt{2}\\ A_0=A_z \qquad A_{\pm1}=\mp(A_x\pm\mathrm{i}A_y)/\sqrt{2}.\end{gathered} \tag{3.3}$$

In the treated case

$$\begin{gathered}A_0=A_z=-\mathrm{i}kgR\sum h_l(kr)P_l(\cos\theta)F_l^0\\ A_{\pm1}=\mp\frac{1}{\sqrt{2}}\exp(\pm\mathrm{i}\varphi)A_\rho\\ A_\rho=\mathrm{i}kgR\sum h_l(kr)P_l^1(\cos\theta)F_l^1.\end{gathered} \tag{3.4}$$

Here

$$F_l^0=\int\mathrm{d}\psi(\cos\psi)g_lP_l \qquad F_l^1=\int\mathrm{d}\psi(\sin\psi)g_lP_l^1. \tag{3.5}$$

In these integrals the argument of g_l is $k\rho$, $\rho=(d^2+R^2+2dR\cos\psi)^{1/2}$, P_l^m are the normalized Legendre polynomials ($Y_l^m(\theta,\varphi)=(-1)^mP_l^m(\cos\theta)\exp(\mathrm{i}m\varphi)/\sqrt{2\pi}$) which have $R\sin\psi/\rho$ as an argument. It follows from (3.5) that only coefficients F_l with even values of l are different from zero. Equations (3.4) are valid outside the sphere of the radius $r=d+R$ (d and R are the solenoid parameters). For $r<d-R$ the role of g_l and h_l in (3.2) should be interchanged. In a static limit equations (3.4) are transformed into

$$\begin{gathered}A_0=-gR\sum_{l=2}^{\infty}\frac{1}{r^{l+1}}\frac{1}{2l+1}P_l(\cos\theta)f_l^0\\ A_\rho=gR\sum_{l=2}^{\infty}\frac{1}{r^{l+1}}\frac{1}{2l+1}P_l^1(\cos\theta)f_l^1.\end{gathered} \tag{3.6}$$

Here

$$f_l^0=\int \mathrm{d}\psi(\cos\psi)\rho^l P_l\left(\frac{R\sin\psi}{\rho}\right)$$

$$f_l^1=\int \mathrm{d}\psi(\sin\psi)\rho^l P_l^1\left(\frac{R\sin\psi}{\rho}\right).$$

For the infinitely thin ($R \ll d$) solenoid these quantities are simplified:

$$f_{2n}=(-1)^n\left(\frac{4n+1}{2}\right)^{1/2}\left(\frac{d}{2}\right)^{2n-1}\binom{2n}{n}\pi nR$$

$$f_{2n}^1=(-1)^n\left(\frac{d}{2}\right)^{2n-1}\pi R\binom{2n-1}{n-1}[n(2n+1)(4n+1)]^{1/2}.$$

For the sake of completeness we consider the case when the current in the solenoid winding exponentially grows or falls ($j\exp(\pm\omega t)$). The non-vanishing cylindrical components of vector potential are

$$A_r=-\frac{gkR}{\sqrt{r}}\exp(\pm\omega t)\sum K_{l+1/2}(kr)P_l(\cos\theta)\int\frac{1}{\sqrt{\rho}}I_{l+1/2}(k\rho)P_l\left(\frac{R\sin\psi}{\rho}\right)\cos\psi\,\mathrm{d}\psi$$

$$A_\rho=\frac{gkR}{\sqrt{r}}\exp(\pm\omega t)\sum K_{l+1/2}(kr)P_l^1(\cos\theta)\int\frac{1}{\sqrt{\rho}}I_{l+1/2}(k\rho)P_l^1\left(\frac{R\sin\psi}{\rho}\right)\sin\psi\,\mathrm{d}\psi. \qquad (3.7)$$

Here I_ν and K_ν are the modified Bessel functions. These equations are valid for $r>d+R$. For $r<d-R$ the role of K and I should be interchanged. These equations may be used for the description of toroidal-like Abrikosov vortices. (See [32] where more complicated integral forms of (3.7) are used.)

So far we have considered the radiation of electromagnetic waves by the time-dependent poloidal current flowing on the surface of toroidal solenoid. The complementary problem is the consideration of electromagnetic oscillations inside the toroidal cavity. Regretfully, the wave equation is not separated in toroidal coordinates. For the toroidal-like waveguide (formed by two coaxial cylinders and two parallel $z=$ constant planes ($\rho_1<\rho<\rho_2$, $z_1<z<z_2$)) the eigenfrequencies and field strengths were obtained in very interesting, yet ancient, references [33]. Further, we have implicitly suggested that the poloidal winding is formed by infinitely thin conductors densely covering the torus surface. Complications arise when these conditions are not satisfied. This is demonstrated in [34] where the electromagnetic waves radiated by the periodical current flowing through the spiral cylindrical winding are studied.

3.2. Main facts on the vector spherical harmonics

Sometimes it is more convenient to represent vector potential and field strengths by means of so-called vector spherical harmonics [21–24]. The easiest way to obtain them is to couple vectorially the spherical unit vectors $\boldsymbol{n}_\mu$ with Y_l^m functions occurring outside the integral sign in the RHS of (3.2):

$$Y_l^m\boldsymbol{n}_{-\mu}=\sum_\lambda C(1,l,-\mu,m;\lambda,m-\mu)\boldsymbol{Y}_{\lambda l}^{m-\mu} \qquad \lambda=l,l\pm1. \qquad (3.8)$$

Here

$$\boldsymbol{Y}_{\lambda l}^m=\sum_\mu C(1l,-\mu,m+\mu;\lambda m)\boldsymbol{n}_{-\mu}Y_l^{m+\mu}$$

are the so called vector spherical harmonics (VSH). Substituting this into (3.2) and combining Clebsch–Gordan coefficients with the Y_l^m functions under the sign of integral we obtain

$$A=\frac{4\pi ik}{c}\sum_{l\lambda m} h_l(kr)\boldsymbol{Y}_{\lambda l}^m\int g_l \boldsymbol{Y}_{\lambda l}^{m*}\boldsymbol{j}\,\mathrm{d}V. \tag{3.9}$$

The VSH $\boldsymbol{Y}_{\lambda l}^m$ are the eigenfunctions of the orbital and total angular momenta and of the third component of the latter (see [22] for details). They are orthonormal on the unit sphere: $\int \boldsymbol{Y}_{\lambda l}^m \boldsymbol{Y}_{\lambda' l'}^{m'*}\,\mathrm{d}\Omega=\delta_{\lambda\lambda'}\delta_{ll'}\delta_{mm'}$. From $\boldsymbol{Y}_{\lambda l}^m$ and the radial functions g_l and h_l one may organize the vector solutions of the free wave equation:

$$\begin{aligned}\boldsymbol{A}_l^m(E)&=(-\sqrt{l}\,h_{l+1}\boldsymbol{Y}_{l,l+1}^m+\sqrt{l+1}\,h_{l-1}\boldsymbol{Y}_{l,l-1}^m)/\sqrt{2l+1}\\ \boldsymbol{A}_l^m(L)&=(\sqrt{l+1}\,h_{l+1}\boldsymbol{Y}_{l,l+1}^m+\sqrt{l}\,h_{l-1}\boldsymbol{Y}_{l,l-1}^m)/\sqrt{2l+1}\qquad \boldsymbol{A}_l^m(M)=-h_l\boldsymbol{Y}_{ll}^m\end{aligned} \tag{3.10}$$

$\boldsymbol{B}_l^m(\tau)$ are obtained from $\boldsymbol{A}_l^m(\tau)$ when the substitution $h_l\to g_l$ is made. The values $\tau=E$, M and L in $\boldsymbol{A}_l^m(\tau)$ correspond to electric, magnetic and longitudinal multipoles. The vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ are the eigenfunctions of the total angular momentum and its third projection. They are orthogonal on the sphere of arbitrary radius. Further, vectors $\boldsymbol{A}$ (or $\boldsymbol{B}$) may be presented in the following alternative form [21–24]

$$\begin{aligned}\boldsymbol{A}_l^m(L)&=\frac{1}{k}\nabla h_l Y_l^m\\ \boldsymbol{A}_l^m(M)&=\frac{1}{\sqrt{l(l+1)}}h_l\boldsymbol{L}Y_l^m\qquad \boldsymbol{L}=-\mathrm{i}\boldsymbol{r}\times\nabla\\ \boldsymbol{A}_l^m(E)&=-\frac{\mathrm{i}}{k}\frac{1}{\sqrt{l(l+1)}}\nabla\times(\boldsymbol{L}h_lY_l^m).\end{aligned} \tag{3.11}$$

Reversing equations (3.10) we may express $h_l\boldsymbol{Y}_{\lambda l}^m(g_l\boldsymbol{Y}_{\lambda l})$ in terms of $\boldsymbol{A}_l^m(\tau)(\boldsymbol{B}_l^m(\tau))$. Substituting them into (3.9) we obtain

$$\boldsymbol{A}=\frac{4\pi ik}{c}\sum \boldsymbol{A}_l^m(\tau)a_l^m(\tau)\qquad a_m^l(\tau)=\int \boldsymbol{B}_l^{m*}(\tau)\boldsymbol{j}\,\mathrm{d}V. \tag{3.12}$$

We refer to $a_l^m(\tau)$ as electrical (E), magnetic (M) and longitudinal (L) form factors. In what follows we need the explicit expressions for $a_l^m(\tau)$. Here they are

$$\begin{aligned}a_l^m(M)&=-J_{ll}^m\\ a_l^m(L)&=(\sqrt{l+1}\,J_{l,l+1}^m+\sqrt{l}\,J_{l,l-1}^m)/\sqrt{2l+1}\\ a_l^m(E)&=(-\sqrt{l}\,J_{l,l+1}^m+\sqrt{l+1}J_{l,l-1}^m)/\sqrt{2l+1}\\ J_{\lambda l}^m&=\int g_l\boldsymbol{Y}_{\lambda l}^*\boldsymbol{j}\,\mathrm{d}V.\end{aligned} \tag{3.13}$$

For $k\to 0$ we obtain

$$J_{\lambda l}\sim k^l\frac{\sqrt{\pi}}{2^{l+1}}\frac{1}{\Gamma(l+\frac{3}{2})}j_{\lambda l}$$

where $j^m_{\lambda l} = \int r^l Y^{m*}_{\lambda l} \boldsymbol{j}\, dV$. This leads to the following asymptotic behaviour of formfactors:

$$a^m_l(M) \sim k^l M^m_l(M) \qquad a^m_l(E) \sim k^{l-1} M^m_l(E)$$
$$a^m_l(L) \sim k^{l-1} M^m_l(L). \tag{3.14}$$

Here $M^m_l(\tau)$ are the E, M, L multipole moments:

$$M^m_l(M) = -\frac{1}{\Gamma(l+\frac{3}{2})} \frac{\sqrt{\pi}}{2^{l+1}} j^m_{ll}$$
$$M^m_l(E) = \sqrt{\frac{\pi(l+1)}{2l+1}} \frac{1}{\Gamma(l+\frac{1}{2})} \frac{1}{2^l} j^m_{l,l-1} \tag{3.15}$$
$$M^m_l(L) = \sqrt{\frac{\pi l}{2l+1}} \frac{1}{\Gamma(l+\frac{1}{2})} \frac{1}{2^l} j^m_{l,l-1}.$$

In the absence of charge density the vector potential (3.12) meets gauge condition div $\boldsymbol{A} = 0$. In this case L formfactors are equal to zero. To prove this we apply the div operator to (3.12). It follows at once from (3.11) that

$$\operatorname{div} \boldsymbol{A}^m_l(M) = \operatorname{div} \boldsymbol{A}^m_l(E) = 0 \qquad \operatorname{div} \boldsymbol{A}^m_l(L) = -k h_l Y^m_l.$$

Thus,

$$\operatorname{div} \boldsymbol{A} = -\frac{1}{c} 4\pi i k^2 \sum h_l Y^m_l a^m_l(L).$$

As the particular terms of this sum are linear independent, so $a^m_l(L) = 0$ if div $\boldsymbol{A} = 0$. Equating $a^m_l(L)$ to zero in equations (3.13) we obtain

$$J^m_{l,l-1} = -\sqrt{\frac{l+1}{l}} J^m_{l,l+1} \qquad a^m_l(E) = -\sqrt{\frac{2l+1}{l}} J^m_{l,l+1}. \tag{3.16}$$

In this case the asymptotic behaviour of $a^m_l(E)$ is modified [29]:

$$a^m_l(E) \sim k^{l+1} M^m_l(E)$$
$$M^m_l(E) = -\sqrt{\frac{2l+1}{l}} \frac{1}{\Gamma(l+\frac{5}{2})} \frac{1}{2^{l+2}} j^m_{l+1,l}. \tag{3.17}$$

3.3. Multipole form factors and moments for the toroidal solenoid

It turns out [29] that for the poloidal current (3.1)

$$a^m_l(M) = a^m_l(L) = 0 \qquad a^m_l(E) = \delta_{m0} a_l(E)$$
$$M^m_l(E) = \delta_{m0} M_l(E)$$

where $a_l(E)$ and $M_l(E)$ are given by

$$a_l(E) = -\frac{gRc}{2\sqrt{2\pi}} \frac{1}{\sqrt{l}} \sqrt{\frac{2l+1}{2l+3}} (\sqrt{l+1}\, F^0_{l+1} - \sqrt{l+2}\, F^l_{l+1}) \tag{3.18}$$

$$M_l(E) = -\frac{1}{\sqrt{\pi}} \frac{Rgc}{\Gamma(l+\frac{3}{2})} \frac{1}{2^{l+2}} \frac{1}{\sqrt{l(l+1)}} \sqrt{\frac{2l+1}{2l+3}} f^0_{l+1} \tag{3.19}$$

(functions F and f were defined earlier ((3.5) and (3.6))). In obtaining (3.19) the following relation was taken into account [29]

$$f_l^1 = -\sqrt{\frac{l+1}{l}} f_l^0. \tag{3.20}$$

Substituting (3.18) and (3.19) into (3.12) we get in the static limit

$$\boldsymbol{A} = Rg\sqrt{2\pi}\sum_{l=1}^{\infty}[(2l+1)(2l+3)]^{-1/2}\frac{1}{r^{l+2}}f_{l+1}^0\boldsymbol{Y}_{l,l+1}. \tag{3.21}$$

It is easy to check that particular components of this equation exactly coincide with (3.6). From (3.21) one easily obtains the field strengths outside the solenoid

$$\boldsymbol{H} = \frac{4\pi k^2}{c}\sum a_l(E)\boldsymbol{A}_l(M) \qquad \boldsymbol{E} = -\frac{4\pi k^2}{c}\sum a_l(E)\boldsymbol{A}_l(E)$$

or in spherical components

$$H_\varphi = \frac{2\sqrt{2\pi}\,\mathrm{i}k^2}{c}\sum h_l P_l^1 a_l(E)$$

$$E_\theta = \frac{2\sqrt{2\pi}\,k^2}{c}\sum\frac{1}{2l+1}[(l+1)h_{l-1} - lh_{l+1}]P_l^1 a_l(E) \tag{3.22}$$

$$E_r = -\frac{2\sqrt{2\pi}\,k^2}{c}\sum\frac{1}{2l+1}[l(l+1)]^{1/2}(h_{l+1}+h_{l-1})P_l a_l(E).$$

The electromagnetic energy flow through the sphere of sufficiently large radius is

$$\frac{1}{2}\frac{1}{4\pi c}\int E_\theta M_\varphi^*\,\mathrm{d}S = \frac{4k^2}{c^3}\sum a_l^2(E).$$

The factor 1/2 in the LHS of this equation takes into account the difference in time dependences of current densities (2.3) and (3.1).

One may wonder what is the profit in presenting vector potential in three different ways ((2.7), (3.4) and (3.12))? There are certain reasons for this. Since (2.7) contains all multipoles in a closed form, it is easier to operate with it in practice. Further, the VSH expansion contains a lot of 'reefs' (discussed in the following section). So, expansion (3.4) may be used as a guiding point to escape them. On the other hand there are physical problems for which the Helmholtz equation is easily solved in terms of VSH, and practically unsolvable when the usual spherical functions are used. The VSH realization of most general non-static solenoid found recently in [35] clearly demonstrates this.

3.4. Toroidal form factors and moments

There is a lot of controversy concerning these quantities (see the discussion in [36]). The usual expansion in terms of VHS is complete. What are the reasons to add something else? The first reason is a physical one. The poloidal current flowing on the torus surface generates a toroidal (or anapole) moment [10, 11, 36] (in exactly the same way as circular current induces a dipole moment) which has clear geometrical sense. It is directed along the torus symmetry axis (figure 1). The second reason is due to the implications arising in the long-wavelength limit. We shall see in this section that

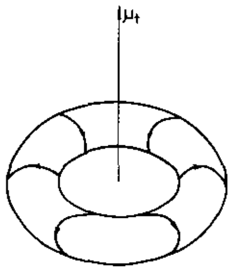

Figure 1. The toroidal moment generated by the poloidal current.

introduction of toroidal form factors makes it possible to separate explicitly singular terms. We have found in the previous section the radiation field of a toroidal solenoid. Based on this we intend now to obtain explicit values of toroidal moments and form-factors and discuss their properties. The usual way [11] to introduce the toroidal moments and form factors starts from the consideration of general case when both charge ($\rho \exp(-\mathrm{i}\omega t)$) and current ($\boldsymbol{j} \exp(-\mathrm{i}\omega t)$) densities are different from zero. The development (3.12) is still valid, but the longitudinal form factors $a_l^m(L)$ are no longer zero. The electric scalar potential is given by

$$\Phi = 4\pi \mathrm{i} k \exp(-\mathrm{i}\omega t) \sum h_l Y_l^m(\theta, \varphi) q_l^m$$

where $q_l^m = \int g_l Y_l^{m*} \rho \, \mathrm{d}V$. Potentials Φ and $\boldsymbol{A}$ now meet the Lorentz gauge condition

$$\operatorname{div} \boldsymbol{A} + \frac{1}{c} \frac{\partial \Phi}{\partial t} = 0$$

from which the relation between q_l^m and $a_l^m(L)$ follows:

$$q_l^m = \frac{\mathrm{i}}{c} a_l^m(L). \tag{3.23}$$

By developing Bessel functions on both sides of this equation we observe that it cannot be satisfied in any order (for non-vanishing charge density). The reason is that ρ and $\boldsymbol{j}$ in (3.23) are connected by the continuity equation $\operatorname{div} \boldsymbol{j} = \mathrm{i}\omega\rho$ and, thus, they are not independent of ω (or k). We consider (3.23) as the definition of q_l^m. Its dependence on k is determined by the Bessel functions in the RHS of (3.23).

We turn again to the expansion (3.12). It is easy to check that the terms relating to E and L multipoles taken separately diverge in the long-wavelength limit. In fact, $a_l^m(E)$ and $a_l^m(L)$ (see equations (3.14)) decrease as k^{l-1} for $k \to 0$ while $A_l^m(E)$ and $A_l^m(L)$ (see equations (3.10)) grow as k^{-l-2}. Taking into account the overall k factor at the front of (3.12), this leads to a k^{-2} divergence for either E or L multipole terms. On the other hand, there are no divergences in (3.2) or (3.9), from which (3.12) easily follows. This means that singularity of E multipole terms is exactly compensated by that of L multipole terms. In fact, substituting $A_l^m(\tau)$ and $a_l^m(\tau)$ into (3.12) and regrouping terms we arrive at (3.9) which does not contain singularities. The other way [11] to deal with singularities in (3.12) is to separate explicitly their contribution to E and L multipole terms.

Combining (3.13) and (3.23) we get

$$a_l^m(E) = -\mathrm{i}c \sqrt{\frac{l+1}{l}} q_l^m - \sqrt{\frac{2l+1}{l}} J_{l,l+1}^m \tag{3.24}$$

or in a slightly different form

$$a_l^m(E)=a_l^m(T)-\mathrm{i}c\sqrt{\frac{l+1}{l}}\,k^{l-1}M_l^m(Q). \tag{3.25}$$

Here $M_l^m(Q)$ is the moment of charge distribution

$$M_l^m(Q)=\frac{i}{c}\frac{1}{2^l}\sqrt{\frac{\pi l}{2l+1}}\frac{1}{\Gamma(l+\frac{1}{2})}j_{l,l-1}^m.$$

It is defined by the asymptotic behaviour of q_l^m for small values of k: $q_l^m\sim k^{l-1}M_l^m(Q)$. The quantity

$$a_l^m(T)=-\mathrm{i}c\sqrt{\frac{l+1}{l}}\,[q_l^m-k^{l-1}M_l^m(Q)]-\sqrt{\frac{2l+1}{l}}\,J_{l,l+1}^m \tag{3.26}$$

is called the toroidal (T) form factor. It decreases as k^{l+1}:

$a_l^m(T)\sim k^{l+1}M_l^m(T)$

$$M_l^m(T)=-\frac{1}{\Gamma(l+\frac{3}{2})}\sqrt{\frac{l+1}{2l+1}}\frac{1}{2^{l+3/2}}\left(\sqrt{\frac{l}{l+1}}\frac{1}{l+\frac{3}{2}}j_{l,l+1}^m+\int r^{l+1}\boldsymbol{Y}_{l,l-1}^{m*}\boldsymbol{j}\,\mathrm{d}V\right). \tag{3.27}$$

This coefficient is referred to as toroidal multipole moment. Substituting $a_l^m(E)$ given by equations (3.24) into expansion (3.12) we observe that singular term in $a_l^m(E)$ involving $M_l^m(Q)$ exactly compensates the singularity of the L multipole term. The toroidal form factor $a_l^m(T)$ appears as a coefficient at $A_l^m(E)$. It gives finite contribution to the vector potential in the long-wavelength limit and, thus, to the coefficient at r^{-l-1} in the solution of the corresponding Laplace equation. Up to now we have considered the general case when both current and charge densities are different from zero. Consider the case of vanishing charge density. As $M_l^m(T)$ contains no sign of charge density ρ it should have the same form of any choice of ρ and, particularly, for $\rho=0$. On the other hand, in the absence of charge density we have $q_l^m=a_l^m(L)=0$ while $a_l^m(E)$ decreases as k^{l+1} (see (3.17)). The terms corresponding to E multipoles in the expansion (3.12) are well-behaved now and no renormalization of $a_l^m(E)$ is needed. So, we simply put $q_l^m=0$ in equations (3.24)-(3.26) and arrive at

$$a_l^m(T)=a_l^m(E)=-\sqrt{\frac{2l+1}{l}}\,J_{l,l+1}^m. \tag{3.28}$$

The E and T multipole moments corresponding to (3.28) are given by (3.17) which is certainly different from $M_l^m(T)$ given by (3.27). To see the reason for this controversy we turn to (3.23). Put $q_l^m=0$ in (3.23) and develop $a_l^m(L)$ in powers of k. The disappearance of coefficients at k^{l-1+2n} results in

$$\sqrt{l}(l+n+\tfrac{1}{2})\int r^{l-1+2n}\boldsymbol{Y}_{l,l-1}^{m*}\boldsymbol{j}\,\mathrm{d}V=n\sqrt{l+1}\int r^{l-1+2n}\boldsymbol{Y}_{l,l+1}^{m*}\boldsymbol{j}\,\mathrm{d}V.$$

For $n=1$ this gives

$$\int r^{l+1}\boldsymbol{Y}_{l,l-1}^{m*}\boldsymbol{j}\,\mathrm{d}V=\sqrt{\frac{l+1}{l}}\frac{1}{l+\frac{3}{2}}j_{l,l+1}^m. \tag{3.29}$$

Multiply (3.29) by an arbitrary constant a and combine it with (3.27) to give

$$M_l^m(T) = -\frac{1}{\Gamma(l+\frac{3}{2})}\sqrt{\frac{l+1}{2l+1}}\frac{1}{2^{l+3/2}}\left[\left(a\sqrt{\frac{l+1}{l}}+\sqrt{\frac{l}{l+1}}\right)\frac{1}{l+\frac{3}{2}}j_{l,l+1}^m + (1-a)\int r^{l+1}\mathbf{Y}_{l,l-1}^{m*}\mathbf{j}\,\mathrm{d}V\right]. \tag{3.30}$$

For $a=1$, $a=0$ and $a=-l/l+1$ we arrive at equations (3.17), (3.27) and

$$M_l^m(T) = -\frac{1}{\Gamma(l+\frac{3}{2})}\sqrt{\frac{2l+1}{l+1}}\frac{1}{2^{l+3/2}}\int r^{l+1}\mathbf{Y}_{l,l-1}^{m*}\mathbf{j}\,\mathrm{d}V \tag{3.31}$$

respectively. Since (3.29) is satisfied identically, so the dependence of a in (3.30) is fictitious. The coincidence of (3.30) with either (3.17) or (3.27) means that (for vanishing charge density) the toroidal form factors and moments are not distinguishable from the electrical ones. For the particular poloidal current (3.1) $a_l^m(T)$ and $M_l^m(T)$ may be written explicitly $a_l^m(T)=\delta_{m0}a_l(E)$, $M_l^m(T)=\delta_{m0}M_l(E)$ where $a_l(E)$ and $M_l(E)$ are defined by (3.18) and (3.19). We thus conclude that poloidal current (3.1) with vanishing charge density radiates only electrical multipoles (more accurately, the multipole expansion of vector potential contains only E multipoles). Toroidal form factors and moments coincide with the electric ones and, thus, for the particular case under consideration they carry no new information. They become important when non-vanishing charge and current densities form non-trivial topological configuration [36].

4. The interaction of a toroidal solenoid with an external electromagnetic field

The interaction of the current $\mathbf{j}$ with the external magnetic field $\mathbf{H}_{\text{ext}}$ is given by

$$U = -\frac{1}{c}\int \mathbf{j}\mathbf{A}\,\mathrm{d}V. \tag{4.1}$$

Here $\mathbf{A}$ is the VP of the magnetic field $\mathbf{H}_{\text{ext}} = \operatorname{rot}\mathbf{A}$. Let the distance between the magnetic field source and the constant current $\mathbf{j}$ flowing in the solenoid's winding be much larger than the solenoid dimensions. Then, in the neighbourhood of the solenoid the vector potential may be presented in the form

$$A_i(\mathbf{r}) = A_i(\mathbf{a}) + \frac{\partial A_i(\mathbf{a})}{\partial x_k}x_k + \frac{1}{2}\frac{\partial^2 A_i}{\partial x_l \partial x_l}x_k x_l + \ldots. \tag{4.2}$$

(It is suggested therefore that $\mathbf{A}$ varies rather slowly in the solenoid's vicinity.) Here $\mathbf{a}$ is some fixed point near the solenoid and $\mathbf{r}$ defines the position of a particular current element with respect to this point. Inserting expansion (4.2) into (4.1) we obtain (for the poloidal current (2.4))

$$U = -\boldsymbol{\mu}_d\mathbf{H}_{\text{ext}} - \tfrac{1}{3}\boldsymbol{\mu}_t \operatorname{rot}\mathbf{H}_{\text{ext}} + \ldots. \tag{4.3}$$

Here $\boldsymbol{\mu}_d$ is the dipole magnetic moment: $\boldsymbol{\mu}_d = \int \mathbf{M}\,\mathrm{d}V$,

$$\boldsymbol{\mu}_t = \int \mathbf{r}\times\mathbf{M}\,\mathrm{d}V \tag{4.4}$$

and $\boldsymbol{M}$ is the magnetic moment density $\boldsymbol{M}=(1/2l)\boldsymbol{r}\times\boldsymbol{j}$. For the poloidal current (3.1) the non-vanishing components of $\boldsymbol{M}$ are:

$$M_x = -M\sin\varphi \qquad M_y = M\cos\varphi$$
$$M=\frac{g}{8\pi c}\,\delta(\tilde{R}-R)\,\frac{R+d\cos\psi}{d+R\cos\psi}. \tag{4.5}$$

This means that only the φ component of $\boldsymbol{M}$ differs from zero ($M_\varphi = M$). It follows from (4.5) that $\boldsymbol{\mu}_d=0$, i.e., the magnetic dipole moment equals zero for the toroidal solenoid. For the poloidal current (3.1) the single nonvanishing component of $\boldsymbol{\mu}_t$ is directed along the symmetry axis of the solenoid (figure 1) and equals

$$\mu_t=\int (xM_y - yM_x)\,\mathrm{d}V=\frac{3}{4c}\,\pi g\,\mathrm{d}R^2. \tag{4.6}$$

Writing out the triple vector product in (4.4) we get

$$\mu_{ti}=\frac{1}{2c}\int [x_i(\boldsymbol{rj})-r^2 j_i]\,\mathrm{d}V=-\frac{3}{4c}\int r^2 j_i\,\mathrm{d}V. \tag{4.7}$$

Comparing (4.7) with (3.31) we recover the coincidence of μ_{ti} with either E or T multipole momenta. Since $\boldsymbol{\mu}_d$ disappears for the toroidal solenoid the interaction (4.1) may be presented in the form

$$U=-\tfrac{1}{3}\boldsymbol{\mu}_t\,\mathrm{rot}\,\boldsymbol{H}_{\mathrm{ext}}. \tag{4.8}$$

Using the Maxwell equation

$$\mathrm{rot}\,\boldsymbol{H}_{\mathrm{ext}}=\frac{1}{c}\frac{\partial \boldsymbol{E}_{\mathrm{ext}}}{\partial t}+\frac{4\pi}{c}\boldsymbol{j}_{\mathrm{ext}}$$

and taking into account that expansion (4.2) is valid at sufficiently large distances from the external field source (where $j_{\mathrm{ext}}=0$) one may rewrite (4.8) as [10, 11, 36]

$$U=-\frac{1}{3c}\boldsymbol{\mu}_t\frac{\partial \boldsymbol{E}_{\mathrm{ext}}}{\partial t}. \tag{4.9}$$

It follows from this that the toroidal solenoid interacts with the external electromagnetic field if the electric field has a non-vanishing and time-dependent component along the symmetry axis of the solenoid. This assertion grounds essentially on the fact that the dimensions of the solenoid are small wrt distance from the electromagnetic field source. Interaction with the static magnetic field is possible if this condition fails. To see this we introduce instead of the current $\boldsymbol{j}$ the magnetization $\mathcal{M}$:

$$c\,\mathrm{rot}\,\boldsymbol{\mathcal{M}}=\boldsymbol{j}. \tag{4.10}$$

The magnetization $\mathcal{M}$ corresponding to $\boldsymbol{j}$ given by (2.4) is

$$\boldsymbol{\mathcal{M}}=\mathcal{M}\boldsymbol{n}_\varphi \qquad \mathcal{M}=\frac{g}{4\pi}\,\frac{\theta(R-\tilde{R})}{d+\tilde{R}\cos\psi}. \tag{4.11}$$

As the current and magnetization formalism are entirely equivalent [21, 37], one may forget about the solenoid current and treat the solenoid as a magnetized ring with magnetization defined by (4.11). Its physical realization [8] is a hard ferromagnetic ring having magnetization (4.11) that is independent of applied fields. Now we substitute (4.10) into (4.1):

$$U = -\int \boldsymbol{A}\,\mathrm{rot}\,\boldsymbol{\mathcal{M}}\,\mathrm{d}V.$$

Integrating this equation by parts one gets

$$U = -\int \boldsymbol{H}_{\mathrm{ext}}\boldsymbol{\mathcal{M}}\,\mathrm{d}V. \tag{4.12}$$

It follows from this that the toroidal solenoid interacts with the external magnetic field, the φ component of which has non-zero overlapping with the solenoid magnetization. As an example, consider the toroidal magnetized ring and the linear current. It turns out that interaction energy (4.12) differs from zero only if the linear current passes through the torus hole.

Consider two toroidal solenoids with constant currents in their windings. Do these solenoids interact? (This question was posed by Smorodinsky [38].) Their interaction is given by

$$-\frac{1}{c^2}\int \frac{\boldsymbol{j}_1(\boldsymbol{r}_1)\boldsymbol{j}_2(\boldsymbol{r}_2)}{|\boldsymbol{r}_1-\boldsymbol{r}_2|}\,\mathrm{d}V_1\,\mathrm{d}V_2 = -\frac{1}{c}\int \boldsymbol{A}_2(\boldsymbol{r})\boldsymbol{j}_1(\boldsymbol{r})\,\mathrm{d}V.$$

Using the relation $\boldsymbol{j}_1 = c\,\mathrm{rot}\,\boldsymbol{\mathcal{M}}$ and integrating by parts one gets

$$-\int \boldsymbol{H}_2(\boldsymbol{r})\boldsymbol{\mathcal{M}}_1(\boldsymbol{r})\,\mathrm{d}V.$$

From this it follows that there is no interaction between the non-overlapping solenoids (as the magnetic strengths and magnetizations are confined inside the solenoids).

5. The model of toroidal solenoid current

It should be explained first why the concrete realization of poloidal current flowing on the solenoid's surface is needed. The reason is that in real conductors (from which the surface current is composed) there are electrons which move relative to the positive lattice ions. There are numerous theoretical considerations (see [25] and references therein) which predict the appearance of electrical field (of the order v^2/c^2) arising from the electron motion. As their drift is of the order 1 mm s^{-1} [27], so these effects are negligible. To increase electron velocity an experiment was performed [28] in which the toroidal solenoid was put into rapid rotation. Although no additional electrical field associated with solenoid rotation was found, there is increasing activity in this field (see [39, 40] and references therein). Our first goal is to evaluate the electromagnetic field (in the framework of usual Maxwell theory) arising from the solenoid's rotation.

Consider torus (2.1). In what follows we shall extensively use the toroidal coordinates

$$x = \frac{a\sinh\mu\cos\varphi}{\cosh\mu - \cos\theta} \qquad y = \frac{a\sinh\mu\sin\varphi}{\cosh\mu-\cos\theta} \qquad z = \frac{a\sin\theta}{\cosh\mu-\cos\theta}$$
$$(0<\mu<\infty \qquad -\pi<\theta<\pi \qquad 0<\varphi<2\pi). \tag{5.1}$$

For μ fixed the points $P(x, y, z)$ fill the surface of the torus with the parameters $d = a \coth \mu$ and $R = a/\sinh \mu$. Let $\mu = \mu_0$ correspond to the torus T. Then, for $\mu > \mu_0$ ($\mu < \mu_0$) the point $P(x, y, z)$ (where x, y, z are given by (5.1)) lies inside (outside) T. The infinitesimal volume element expressed in toroidal coordinates is

$$dV = a^3 \frac{\sinh \mu \, d\mu \, d\theta \, d\varphi}{(\cosh \mu - \cos \theta)^3}.$$

The current density $\boldsymbol{j}$ is given by

$$\boldsymbol{j} = -\frac{gc}{4\pi a^2} \frac{1}{\sinh \mu_0} \delta(\mu - \mu_0)(\cosh \mu_0 - \cos \theta)^2 \cdot \boldsymbol{n}_\theta. \tag{5.2}$$

Here $\boldsymbol{n}_\theta$ is the same as $\boldsymbol{n}_\psi$ (see section 2) but being expressed in toroidal coordinates:

$$\boldsymbol{n}_\theta = [(\boldsymbol{n}_x \cos \varphi + \boldsymbol{n}_y \sin \varphi) \sinh \mu \sin \theta + \boldsymbol{n}_z(1 - \cosh \mu \cos \theta)](\cosh \mu - \cos \theta)^{-1}.$$

For constant current $\boldsymbol{H} = \boldsymbol{n}_\varphi \cdot g/\rho$ inside the solenoid and $\boldsymbol{H} = 0$ outside it. The VP has two non-vanishing cylindrical components (A_ρ and A_z). At large distances they decrease as r^{-3}

$$A_z \sim \tfrac{1}{8}\pi g a^3 \frac{\cosh \mu_0}{\sinh^3 \mu_0} \frac{1 + 3\cos 2\theta_s}{r^3} \qquad A_\rho \sim \tfrac{3}{8}\pi g a^3 \frac{\cosh \mu_0}{\sinh^3 \mu_0} \frac{\sin 2\theta_s}{r^3}$$

(θ_s is the spherical polar angle).

Their explicit expressions are given in [19]. Now we try to simulate the solenoid current as a relative motion of charged layers, out of which the surface charge distribution is composed. We require the following conditions to be fulfilled: (1) the total charge should be equal zero; (2) the electrostatic potential should vanish both inside and outside the solenoid; (3) the rotation of the elements composing the particular charged layer in the φ = constant plane (each element rotates in that φ = constant plane in which it lies) should reproduce the current distribution (5.2) and as a consequence the vector potential of the toroidal solenoid.

We seek the charge distribution in the form

$$\sigma = \sigma_1 \delta(\mu - \mu_1) + \sigma_2 \delta(\mu - \mu_2) + \sigma_0 \delta(\mu - \mu_0). \tag{5.3}$$

For definiteness we choose $\mu_1 > \mu_2 > \mu_0$. The charge distribution consists of three charged toroidal shells encompassing each other (it is impossible to meet conditions (1)-(3) using a charge distribution consisting of two layers). The layers corresponding to $\mu = \mu_0$ and $\mu = \mu_1$ are external and internal ones (figure 2). Let each element of the

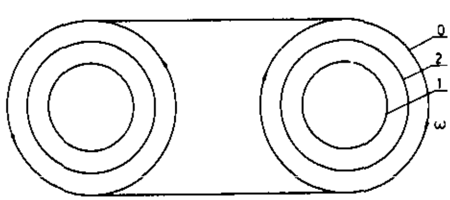

Figure 2. The model of poloidal current. Charge distribution consists of three charged shells (0, 1, 2). The electrostatic potential differs from zero only between shells 0 and 1. The rotation of external shell 0 simulates poloidal current.

external layer ($\mu=\mu_0$) rotate in the $\varphi=$ constant plane with angular velocity ω. We choose σ_0 so as to reproduce the current density (5.2). This gives

$$\sigma_0\omega R=-\frac{gc}{4\pi a^2}\frac{(\cosh\mu_0-\cos\theta)^2}{\sinh\mu_0}. \tag{5.4}$$

Here $R=a/\sinh\mu_0$ is the radius of the external shell. Or in a slightly different form

$$\sigma_0=f\frac{(\cosh\mu_0-\cos\theta)^2}{\sinh\mu_0} \qquad f=-gc\frac{\sinh\mu_0}{4\pi a^2\omega}. \tag{5.5}$$

We give only the final answer. The electrostatic potential vanishing both inside ($\mu_1<\mu<\infty$) and outside ($0<\mu<\mu_0$) solenoid equals

$$\Phi=8\sqrt{2}fa^2(\cosh\mu-\cos\theta)^{1/2}\sum\frac{1}{1+\delta_{n0}}Q_{n-1/2}(0)\cos n\theta$$
$$\times[P_{n-1/2}(0)Q_{n-1/2}-P_{n-1/2}Q_{n-1/2}(0)]$$

between '0' and '2' charged shells ($\mu_0<\mu<\mu_2$) and

$$\Phi=8\sqrt{2}fa^2(\cosh\mu-\cos\theta)^{1/2}\sum\frac{1}{1+\delta_{n0}}Q_{n-1/2}(0)\frac{r_n(0,2)}{r_n(1,2)}\cos n\theta$$
$$\times[P_{n-1/2}(1)Q_{n-1/2}-P_{n-1/2}Q_{n-1/2}(1)]$$

between '1' and '2' charged shells ($\mu<\mu<\mu_1$). Here we put

$$P_\nu^\lambda(i)\equiv P_\nu^\lambda(\cosh\mu_i) \qquad Q_\nu^\lambda(i)\equiv Q_\nu^\lambda(\cosh\mu_i) \qquad i=0,1,2$$
$$r_n(i,j)=P_{n-1/2}(i)Q_{n-1/2}(j)-Q_{n-1/2}(i)P_{n-1/2}(j).$$

From now we do not indicate the argument of the Legendre functions if it equals $\cosh\mu$.

The discontinuity of $\mathrm{d}\Phi/\mathrm{d}\mu$ at $\mu=\mu_1$ and $\mu=\mu_2$ fixes σ_1 and σ_2

$$\sigma_1=-\frac{2\sqrt{2}}{\pi}f\frac{(\cosh\mu_1-\cos\theta)^{5/2}}{\sinh\mu_1}\sum\frac{\cos n\theta}{1+\delta_{n0}}Q_{n-1/2}(0)\frac{r_n(0,2)}{r_n(1,2)}$$
$$\sigma_2=-\frac{2\sqrt{2}}{\pi}f\frac{(\cosh\mu_1-\cos\theta)^{5/2}}{\sinh\mu_2}\sum\frac{\cos n\theta}{1+\delta_{n0}}Q_{n-1/2}(0)\frac{r_n(1,0)}{r_n(1,2)}.$$

The constant f determines the value of the charge on each of the layers 0, 1 and 2:

$$e_0=\int\sigma_0\delta(\mu-\mu_0)\,\mathrm{d}V=4\pi^2fa^3/\sinh\mu_0$$
$$e_1=\int\sigma_1\delta(\mu-\mu_1)\,\mathrm{d}V=-16a^3f\sum\frac{1}{1+\delta_{n0}}Q_{n-1/2}(0)Q_{n-1/2}(1)\frac{r_n(0,2)}{r_n(1,2)}$$
$$e_2=\int\sigma_2\,\delta(\mu-\mu_2)\,\mathrm{d}V=-16a^3f\sum\frac{1}{1+\delta_{n0}}Q_{n-1/2}Q_{n-1/2}(2)\frac{r_n(1,0)}{r_n(1,2)}.$$

It is easy to check that

$$e_0+e_1+e_2=0$$

i.e. the treated charged distribution is electrically neutral. The uniform rotation of the elements composing '0' shell in the φ = constant plane with angular velocity ω simulates the surface current (5.2). As a result, the vector potential satisfying the Poisson equation

$$\Delta A_\omega=-\frac{4\pi}{c}j_\omega$$

is generated. The constant g entering into the definition of $\boldsymbol{j}$ (see (5.2)) may be expressed through

$$g = -\omega e_0/\pi c.$$

This means that for $\boldsymbol{A}_\omega$ we may use explicit expressions of the VP obtained in [19] with g defined by the last equation. Thus the electrostatic potential Φ and surface densities σ_i meet conditions (1)-(3) mentioned above.

6. The rotating toroidal solenoid

In what follows we consider the rotation of the toroidal solenoid with current distribution constructed in a previous section. The charge density of this solenoid differs from zero (it consists of three charged shells). This means that the electromagnetic field generated by its rotation should not coincide with the field of the rotated magnetized ring with the magnetization (4.11) for which charge density equals zero. Let the solenoid rotate as a whole around the symmetry axis z with angular velocity Ω (figure 3). Then in addition to the poloidal current (5.2) there appears current

$$\boldsymbol{j}_\Omega = j_\varphi^\Omega \boldsymbol{n}_\varphi \qquad j_\varphi^\Omega = \sigma v_\varphi = \Omega\rho\sigma \tag{6.1}$$

flowing in the latitude direction. Here σ is given by (5.3), $\rho = a \sinh\mu/(\cosh\mu - \cos\theta)$ is the distance between a particular element of the charge shell and the z axis. The current (6.1) generates VP satisfying the Poisson equation $\Delta\boldsymbol{A}_\Omega = -(4\pi/c)\boldsymbol{j}_\Omega$. It turns out that $\boldsymbol{A}_\Omega$ has the single non-vanishing component (A_φ^Ω) given by

$$A_\varphi^\Omega = (\cosh\mu - \cos\theta)^{1/2} \sum \frac{1}{1+\delta_{n0}} A_n(\mu) \cos n\theta. \tag{6.2}$$

The functions $A_n(\mu)$ are given in [29]. For the infinitely thin charge distribution $(\mu_1 = \mu_0 + \Delta_1,\ \mu_2 = \mu_0 + \Delta_1,\ \Delta_1 \ll \mu_0,\ \Delta_2 \ll \mu_0)$ they are equal to

$$A_n = \alpha_n [Q_{n-1/2}^1(0)]^2 P_{n-1/2}^1$$

outside the solenoid

$$A_n = \alpha_n Q_{n-1/2}^1(0) P_{n-1/2}^1(0) Q_{n-1/2}^1$$

inside it. Here

$$\alpha_n = \frac{8\sqrt{2} f a^3}{c(n^2 - 1/4)} \Omega \coth\mu_0(\Delta_1 + \Delta_2).$$

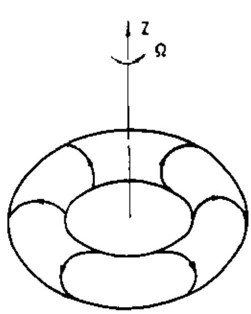

Figure 3. The uniform rotation of solenoid around its symmetry axis leads to the appearance of a magnetic field outside the solenoid T. The electric field arises for non-uniform rotation.

At large distances A_φ^Ω decreases as r^{-2}, i.e. $A_\varphi^\Omega \sim d\Omega \sin\theta_s/r^2$. Here θ_s is the polar angle. For the infinitely thin charge layer (general case is considered in [29]) constant α is

$$\alpha = \frac{e_0}{2\sqrt{2}\,c} \coth\mu_0 \times (\Delta_1 + \Delta_2)\left(1 + \frac{6}{x_0} + \frac{6}{x_0^2}\right) \qquad x_0 = \exp(2\mu_0) - 1.$$

The non-vanishing components of magnetic field decrease as r^{-3}

$$H_r \sim \frac{2\alpha\Omega \cos\theta_s}{r^3} \qquad H_\theta \sim \frac{d\Omega \sin\theta_s}{r^3}. \tag{6.3}$$

The electric field $\boldsymbol{E} = 0$. The total vector potential of a rotating solenoid is

$$\boldsymbol{A} = \boldsymbol{A}_\omega + \boldsymbol{A}_\Omega$$

where $\boldsymbol{A}_\omega$ is the vector potential of the stationary solenoid and $\boldsymbol{A}_\Omega = A_\varphi^\Omega \boldsymbol{n}_\varphi$.

It follows from this that there is no electric field outside the uniformly rotating toroidal solenoid. This is confirmed experimentally [28].

Let the solenoid rotate as a whole around its symmetry axis with the velocity linearly growing with time: $v = \Omega p t$. Then the total vector potential turns out to be equal to

$$\boldsymbol{A} = \boldsymbol{A}_\omega + \left(tA_\varphi^\Omega - \frac{1}{c}\beta\right)\boldsymbol{n}_\varphi. \tag{6.4}$$

Here $\boldsymbol{A}_\omega$ and A_φ^Ω were defined above. The constant β is also given in [29]. As a result, the constant electric field $E_\varphi = -1/cA_\varphi^\Omega$ and the linearly increasing magnetic field $\boldsymbol{H} = t\,\mathrm{rot}(A_\varphi^\Omega \cdot \boldsymbol{n}_\varphi)$ arises outside the solenoid. At large distances one has

$$E_\varphi \sim -\frac{1}{c}\frac{\alpha\Omega \sin\theta_s}{r^2} \qquad H_r \sim \frac{2\alpha\Omega t \cos\theta_s}{r^3} \qquad H_\theta \sim \frac{\alpha\Omega t \sin\theta_s}{r^3}. \tag{6.5}$$

The radial component of the Poynting vector is directed off the solenoid:

$$S_2 = \frac{t\alpha^2\Omega^2 \sin^2\theta_s}{4\pi c^2 r^5}.$$

We thus conclude that the closed current arising from the accelerated rotation of the solenoid leads to the appearance of an electric field outside that solenoid. This does not contradict the Maxwell equations. In fact, field strengths (6.5) are their direct consequence.

7. Conclusion

We briefly summarize the main results obtained:

(1) The electromagnetic radiation field of the toroidal solenoid is obtained. Its properties are investigated.

(2) The multipole expansion of the solenoid's radiation field is obtained. The multipole form factors and moments are evaluated and their relation to the toroidal ones are established.

(3) It is shown how the toroidal solenoid interacts with an external electromagnetic field.

(4) It is shown that a magnetic field arises outside a uniformly rotating toroidal solenoid. Both electric and magnetic fields appear outside a non-uniformly rotating solenoid.

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